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THE AFFINE SCHEME ASSOCIATED TO A NONNOETHERIAN PRIME PI ALGEBRA

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ABSTRACT
It is shown how abstract localization theory may be applied in order to associate to a not necessarily noetherian pi algebra a ringed space (Spec(R), O_R), which behaves functorially with respect to extensions and which possesses suitable features allowing one to study this type of ring from a geometric point of view. These results generalize previous ones, obtained by F. Van Oystaeyen and the author in the noetherian case.

In [13] it was shown how to associate to any left noetherian prime pi algebra R a noncommutative "affine scheme" (Spec(R), O_R). Here Spec(R) is the space of all prime ideals of R and O_R is a structure sheaf on Spec(R), which is constructed locally by using a combination of symmetric localization as introduced by D. Murdoch and F. Van Oystaeyen in [8, 11] and bimodule localization as studied by F. Van Oystaeyen and A. Verschoren in [13]. It was also pointed out in [13] how to construct a similar structure sheaf in the prime case, even if R is not necessarily left noetherian. The latter construction however is only useful in studying birationality questions, since it does not possess nice enough functorial features in general, mainly because it is not defined by means of localization techniques. In the present note, we will show how to construct structure sheaves in the prime case, that behave functorially even in the absence of the noetherian hypothesis. If the base ring R is commutative, then the sheaf we construct coincides with the usual structure sheaf on Spec(R). Moreover, if R is not necessarily commutative, then this sheaf coincides with the one considered above, whenever R is left noetherian. Finally, the ringed space we thus obtain behaves functorially with respect to extensions in the sence of C. Procesi [9] and

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yields back the ring R as the ring of global sections. Briefly: the structure sheaf constructed this way enjoys the same properties as the one studied in [13], but "works" also in the nonnoetherian case.

1. For simplicity's sake we will assume throughout that $R, \dots$ is an affine prime pi algebra over a field. We assume the reader to be familiar with the language and main results of the theory of localization at an idempotent kernel functor in R-mod, the category of left R-modules, such as exposed in [5, 6, 10]. Recall that if σ is an idempotent kernel functor in R-mod, then we denote by Q_σ the associated localization function and by $\mathscr{L}(\sigma)$ the associated Gabriel filter. The quotient category at σ is denoted by (R,σ)-mod; it consists of all R-modules M which are σ-closed, i.e. such that the natural morphism $j_\sigma : M \to Q_\sigma(M)$ is an isomorphism. When R is left noetherian, typical examples of idempotent kernel functors in R-mod are σ_{R-P} and σ_I, where $P \in \mathrm{Spec}(R)$ and I is a (two sided!) ideal of R, defined by their Gabriel filters $\mathscr{L}(\sigma_{R-P})$ resp. $\mathscr{L}(\sigma_I)$, which consist of all left ideals L of R such that there exists an ideal $I \not\subset P$ with $I \subset L$ resp. such that there exists a positive integer n with $I^n \subset L$. If R is not left noetherian, then in general these definitions do not yield Gabriel filters, as one easily verifies. However, if R is commutative, then this construction works for any prime ideal P and any finitely generated ideal I of R.

2. Define an idempotent kernel functor σ^I in R-mod by its torsion free class $\mathscr{F}_I$ which consists of all $M \in R$-mod such that $\mathrm{Ann}_M(I) = \{m \in M; Im = 0\} = 0$. It is easily verified that this yields a torsionfree class, indeed. Moreover, we have $L \in \mathscr{L}(\sigma^I)$ if and only if R/L is σ^I-torsion, i.e. $0 = \mathrm{Hom}_R(R/L, M) = \mathrm{Ann}_M(L)$ for all $M \in \mathscr{F}_I$, i.e. $\mathrm{Ann}_M(L) = 0$, whenever $\mathrm{Ann}_M(I) = 0$ for any $M \in R$-mod. It follows that the quotient category (R,σ^I)-mod consists of all $M \in R$-mod such that the canonical map $M = \mathrm{Hom}_R(R, M) \to \mathrm{Hom}_R(I, M)$ is bijective. This idempotent kernel functor was first considered by B. Mueller in [7].

3. LEMMA. *If R is left noetherian, then for any ideal I of R, we have $\sigma^I = \sigma_I$.*

PROOF. If M is σ_I-torsionfree, then in particular, for any $m \in M$, we have that $Im = 0$ implies $m = 0$, hence $\mathrm{Ann}_M(I) = 0$ and $M \in \mathscr{F}_I$. Conversely, if M is σ^I-torsionfree and $m \in \sigma_I M$, then $I^n m = 0$ for some positive integer n, so $I(I^{n-1}m) = 0$ implies that $I^{n-1}m = 0$, as $\mathrm{Ann}_M(I) = 0$, hence by iteration we obtain $m = 0$. It follows that M is σ_I-torsionfree. As the torsionfree classes of σ^I and σ_I coincide, we get $\sigma^I = \sigma_I$. □

4. Lemma. *For any ideal I of R we have $\sigma^I = \sigma^{\mathrm{rad}(I)}$.*

Proof. It is clear that if J and K are ideals of R with $K \in \mathscr{L}(\sigma^J)$, then $\sigma^K \leqq \sigma^J$, hence $\sigma^{\mathrm{rad}(I)} \leqq \sigma^I$. Conversely, from A. Braun's [2] it follows that for any ideal I of R we may find a positive integer n such that $\mathrm{rad}(I)^n \subset I$, hence $I \in \mathscr{L}(\sigma^{\mathrm{rad}(I)})$, and $\sigma^I \leqq \sigma^{\mathrm{rad}(I)}$. Indeed, since R/I is affine over a field, its Jacobson radical is nilpotent by [2]. But R/I is also a Hilbert algebra (cf. [9]), hence its Jacobson radical coincides with its prime radical. As $\mathrm{rad}(R/I) = \mathrm{rad}(I)/I$, this yields the assertion. □

5. Recall that an R-bimodule (in the sense of M. Artin [1]) is a two sided R-module M which is generated over R by its R-centralizer M^R, which consists of all $m \in M$ such that $rm = mr$ for all $r \in R$. Similarly, a ring morphism $f: R \to S$ is said to be an extension (in the sense of C. Procesi [9]) if f endows S with an R-bimodule structure, i.e. $S = f(R)S^R$, where $S^R = \{s \in S; f(r)s = sf(r)$ for all $r \in R\}$. Although the category $\mathrm{bi}(R)$ of all R-bimodules (considered as a full subcategory of R-mod-R, the category of two-sided R-modules) is not even abelian in general, a localization theory in $\mathrm{bi}(R)$ may be developed as in the one-sided case. In particular, an idempotent kernel functor in $\mathrm{bi}(R)$ is a left exact subfunctor σ of the inclusion $\mathrm{bi}(R) \to R$-mod-R, such that $\sigma(M/\sigma M) = 0$ for any R-bimodule M. Note that kernels, cokernels and exactness properties will always be considered within R-mod-R. The bimodule of quotients of an R-bimodule M at σ is by definition an R-bimodule morphism $j_\sigma : M \to Q_\sigma^{\mathrm{bi}}(M)$ such that $\mathrm{Ker}(j_\sigma)$ and $\mathrm{Coker}(j_\sigma)$ are σ-torsion and such that $Q_\sigma^{\mathrm{bi}}(M)$ is faithfully σ-injective, i.e. such that for any exact sequence $0 \to E' \to E \to E'' \to 0$ of R-bimodules such that E'' is σ-torsion and any $f \in \mathrm{Hom}_{\mathrm{bi}(R)}(E', Q_\sigma^{\mathrm{bi}}(M))$ there exists a unique $g \in \mathrm{Hom}_{\mathrm{bi}(R)}(E, Q_\sigma^{\mathrm{bi}}(M))$ extending f. We have proved in [13] that such a bimodule of quotients always exists and that it is essentially unique. It has also been pointed out, cf. [13, IV. 1.29], that an R-bimodule M is σ-injective in $\mathrm{bi}(R)$ if and only if for all $I \in \mathscr{L}^2(\sigma)$, the set of all ideals I of R such that R/I is σ-torsion, and any $f: I \to M$ in R-mod-R, there exists $q \in M^R$ such that $f(i) = iq$ for all $i \in I$.

One should be careful, however, that the filter $\mathscr{L}^2(\sigma)$ does not determine σ unambiguously. Yet, if $m \in M^R$, then $m \in \sigma M$ if and only if $Im = 0$ for some ideal $I \in \mathscr{L}^2(\sigma)$.

6. It is clear that any idempotent kernel functor σ in R-mod induces (by restriction) an idempotent kernel functor in $\mathrm{bi}(R)$, hence for any such σ we may

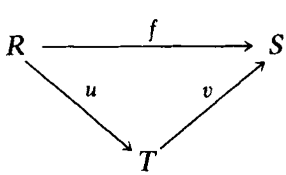
construct the R-bimodule of quotients $Q_\sigma^{\mathrm{bi}}(M)$. This bimodule may be obtained as follows: from [13, IV. 1.22] one deduces that the localization $Q_\sigma(M)$ of M in R-mod may be endowed with a canonical, essentially unique two-sided R-module structure and an easy verification shows that we may put $Q_\sigma^{\mathrm{bi}}(M) = RQ_\sigma(M)^R$. In particular, $Q_\sigma^{\mathrm{bi}}(R)$ is a ring, the canonical morphism $j_\sigma : R \to Q_\sigma^{\mathrm{bi}}(R)$ is an extension and $Q_\sigma^{\mathrm{bi}}(M)$ is a $Q_\sigma^{\mathrm{bi}}(R)$-bimodule for any $M \in \mathrm{bi}(R)$. Of course, if R is commutative, then we have $Q_\sigma^{\mathrm{bi}}(M) = Q_\sigma(M)$ for any $M \in R\text{-mod} = \mathrm{bi}(R)$.

Let us denote by $\mathrm{Spec}(R)$ the space of prime ideals of R endowed with the Zariski topology, i.e. the open subsets of $\mathrm{Spec}(R)$ are of the form $X(I) = \{P \in \mathrm{Spec}(R); I \not\subset P\}$ for some ideal I of R. Note that $X(I) = X(\mathrm{rad}(I))$. If we write $Q_I^{\mathrm{bi}}(M)$ for the R-bimodule $Q_{\sigma^I}^{\mathrm{bi}}(M)$, then we obtain:

7. Proposition. *Associating $Q_I^{\mathrm{bi}}(R)$ to the open set $X(I) \subset \mathrm{Spec}(R)$ defines a presheaf of rings on* $\mathrm{Spec}(R)$. *If we denote by $\mathcal{O}_R$ the associated sheaf on* $\mathrm{Spec}(R)$, *then (for affine prime pi algebras R) the "affine scheme"* $(\mathrm{Spec}(R), \mathcal{O}_R)$ *behaves functorially with respect to ring extensions in the sense of Procesi.*

Proof. If $X(I) \subset X(J)$, then $\mathrm{rad}(I) \subset \mathrm{rad}(J)$, hence $\mathcal{F}_{\mathrm{rad}(I)} \subset \mathcal{F}_{\mathrm{rad}(J)}$, so $\sigma^I \geqq \sigma^J$, by Lemma 4, and we obtain an essentially unique ring morphism $\rho(I,J): Q_J^{\mathrm{bi}}(R) \to Q_I^{\mathrm{bi}}(R)$, which induces the identity on R, by [13, IV. 1.37.]. This shows immediately that we thus obtain a presheaf of rings $\mathcal{O}_R$ on $\mathrm{Spec}(R)$. If $f: R \to S$ is an extension, then it induces a continuous morphism $\phi = {}^a f : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ by sending $Q \in \mathrm{Spec}(S)$ to $f^{-1}(P) \in \mathrm{Spec}(R)$ and if $X(I)$ is an open subset of $\mathrm{Spec}(R)$, then $\phi^{-1}(X(I)) = X(Sf(I)) \subset \mathrm{Spec}(S)$. Note that $Sf(I)$ is an ideal of S, since f is an extension, cf. [9].

Let us now show that for any f as above and any ideal I of R there exists a unique ring morphism $f_I : Q_I^{\mathrm{bi}}(R) \to Q_{Sf(I)}^{\mathrm{bi}}(S)$ extending f. First factorize f through $T = R/\mathrm{Ker}(f)$ as follows:



Here u is a central extension, i.e. $T = u(R)Z(T)$, as u is surjective and v is an extension, since f is an extension. Moreover, T is again an affine prime pi algebra.

Let us construct f_I in 3 steps.

(a) $Q_I^{\mathrm{bi}}(R) \to Q_{u_*\sigma^I}^{\mathrm{bi}}(T)$

The idempotent kernel function σ^I induces an idempotent kernel functor $u_*\sigma^I$ in T-mod, with torsion class given by the left T-modules M such that ${}_RM$, the left R-module from M by restriction of scalars through u, is σ-torsion. Of course, $u_*\sigma^I$ induces an idempotent kernel functor in bi(T), denoted in the same way, and from [13, IV.2.5] and the fact that u is a central extension, it follows that there exists a canonical ring isomorphism $Q_I^{\mathrm{bi}}({}_RT) \simeq Q_{u_*\sigma^I}^{\mathrm{bi}}(T)$ and hence a central extension $Q_I^{\mathrm{bi}}(R) \to Q_{u_*\sigma^I}^{\mathrm{bi}}(T)$.

(b) $Q_{u_*\sigma^I}^{\mathrm{bi}}(T) \to Q_{u(I)}^{\mathrm{bi}}(T)$

First note that if $g : U \to V$ is an arbitrary extension and K is an ideal of U, then for all $L \in \mathscr{L}(\sigma^K)$ we have $Vg(L) \in \mathscr{L}(\sigma^{Vg(K)})$. Indeed, if $N \in V$-mod has the property that $\mathrm{Ann}_N(Vg(K)) = 0$, then, if $Vg(L)n = 0$ for some $n \in N$, we have that $n \in \mathrm{Ann}_{U^N}(L) = 0$, since $\mathrm{Ann}_{U^N}(K) = 0$. It follows that $Vg(L) \in \mathscr{L}(\sigma^{Vg(K)})$. Now, if $L \in \mathscr{L}(u_*\sigma^I)$, then T/L is σ^I-torsion by definition, so $u(L') \subset L$ for some $L' \in \mathscr{L}(\sigma^I)$. But $u(L') = Tu(L') \in \mathscr{L}(\sigma^{u(I)})$ by the foregoing, hence $L \in \mathscr{L}(\sigma^{u(I)})$. It follows that $u_*\sigma^I \leqq \sigma^{u(I)}$, so [13, IV.I.37] yields the existence of an extension $Q_{u_*\sigma^I}^{\mathrm{bi}} \to Q_{u(I)}^{\mathrm{bi}}(T)$.

(c) $Q_{u(I)}^{\mathrm{bi}}(T) \to Q_{Sf(I)}^{\mathrm{bi}}(S)$

Since $v : T \to S$ is an injective extension of prime pi algebras, v induces an injective extension $v' : Q(T) \to Q(S)$ between their (central simple) classical rings of fractions. Since T and S are prime, they are clearly $\sigma^{u(I)}$- resp. $\sigma^{Sf(I)}$-torsionfree, hence for any $q \in Q_{u(I)}(T)$ we may find a left ideal $L \in \mathscr{L}(\sigma^{u(I)})$ such that $Lq \subset T$. It follows that $Sv(L)v'(q) \subset S$ and since $Sv(L) \in \mathscr{L}(\sigma^{Sv(u(I))}) = \mathscr{L}(\sigma^{Sf(I)})$ by the remarks made in (b), we obtain that v' maps $Q_{u(I)}(T)$ into $Q_{Sf(I)}(S)$. Since one easily checks that this map sends $Q_{u(I)}(T)^T \subset Z(Q(T))$ into $Q_{Sf(I)}(S)^S \subset Z(Q(S))$, we get an extension $Q_{u(I)}^{\mathrm{bi}}(T) \to Q_{Sf(I)}^{\mathrm{bi}}(S)$. The composition of the previous maps yields an extension $Q_I^{\mathrm{bi}}(R) \to Q_{Sf(I)}^{\mathrm{bi}}(S)$, which extends $f : R \to S$. Assume that $g_1, g_2 : Q_I^{\mathrm{bi}}(R) \to Q_{Sf(I)}^{\mathrm{bi}}(S)$ both extend f, then $h = g_1 - g_2$ is an R-bimodule morphism which factorizes through $\mathrm{Coker}(j_I : R \to Q_I^{\mathrm{bi}}(R))$. The image of the induced morphism $\bar{h} : \mathrm{Coker}(j_I) \to Q_{Sf(I)}^{\mathrm{bi}}(S)$ is a σ^I-torsion R-bimodule. On the other hand, since $f_*\sigma^I \leqq \sigma^{Sf(I)}$ by the remarks made in (b) and since $Q_{Sf(I)}^{\mathrm{bi}}(S)$ is $\sigma^{Sf(I)}$-torsionfree, we obtain that $Q_{Sf(I)}^{\mathrm{bi}}(S)$ is a σ^I-torsionfree R-module. It follows that $\bar{h}$ is the zero-morphism and that $g_1 = g_2$. For any open $X(I) \subset \mathrm{Spec}(R)$, let $\theta_1(X(I)) : \boldsymbol{Q}_R(X(I)) \to (\phi_*\boldsymbol{Q}_S)(X(I))$ be the map

$$f_I : \boldsymbol{Q}_R(X(I)) = Q_I^{\mathrm{bi}}(R) \to Q_{Sf(I)}^{\mathrm{bi}}(S)$$
$$= \boldsymbol{Q}_S(X(Sf(I))) = \boldsymbol{Q}_S(\phi^{-1}(X(I))) = (\phi_*\boldsymbol{Q}_S)(X(I))$$

then one easily verifies that the collection $\{\theta(X(I))\}$ defines a morphism of presheaves of rings $\theta_1 : \mathbf{Q}_R \to \phi_* \mathbf{Q}_S$ hence of sheaves of rings $\theta : \mathbf{O}_R \to \phi_* \mathbf{O}_S$ (by sheafication). This yields a morphism of ringed spaces $(\mathrm{Spec}(S), \mathbf{O}_S) \to (\mathrm{Spec}(R), \mathbf{O}_R)$, which may be checked to define a contravariant functor $(\mathrm{Spec}(R), \mathbf{O}_R)$ with respect to ring extensions. We leave details to the reader. □

8. *Note.* From Lemma 3, it follows that the ringed space $(\mathrm{Spec}(R), \mathbf{O}_R)$ coincides with the one constructed in [13], whenever R is left noetherian.

9. Let us now calculate the stalks of the sheaf $\mathbf{O}_R$ or, equivalently, of the presheaf $\mathbf{Q}_R$. As we have pointed out before, since R is prime, it is torsionfree for all σ^I hence all "restriction morphisms" $\rho(I,J): Q_J^{\mathrm{bi}}(R) \to Q_I^{\mathrm{bi}}(R)$ are actually injective, so the presheaf $\mathbf{Q}_R$ is separated. It follows for any $P \in \mathrm{Spec}(R)$ that

$$\mathbf{O}_{R,P} = Q_{R,P} = \varinjlim_{X(I) \ni P} Q_I^{\mathrm{bi}}(R) = \bigcup_{I \not\subset P} Q_I^{\mathrm{bi}}(R) \subset Q(R).$$

Recall that for any prime ideal P of R we may define an idempotent kernel functor σ_p in $\mathrm{bi}(R)$ by putting

$$\sigma_P(M) = \bigcap \{\mathrm{Ker}(f); f \in \mathrm{Hom}_{\mathrm{bi}(R)}(M, E^{\mathrm{bi}}(R/P))\}$$

for any R-bimodule M. Here $E^{\mathrm{bi}}(R/P)$ is the injective hull (in $\mathrm{bi}(R)$) of R/P. Actually, σ_P is induced by an idempotent kernel functor in R-mod-R, defined in a similar way, but replacing $E^{\mathrm{bi}}(R/P)$ by $E(R/P)$, the injective hull of R/P in R-mod-R. It is easy to see that $\mathcal{L}^2(\sigma_p)$ consists of all ideals I of R with $I \not\subset P$. Moreover, if $\mathcal{L}(\sigma_{R-P})$ defined in **1** is a Gabriel filter (e.g. if R is left noetherian or commutative) then $Q_{\sigma_P}^{\mathrm{bi}}(R) = Q_{\sigma_{R-P}}^{\mathrm{bi}}(R)$, cf. [13. IV.3.4]. Let us write R_P for $Q_{\sigma_P}^{\mathrm{bi}}(R)$.

10. Proposition. *For any $P \in \mathrm{Spec}(R)$ we have $\mathbf{O}_{R,P} = R_P$.*

Proof. We have already pointed out that

$$\mathbf{O}_{R,P} = \bigcup_{I \not\subset P} Q_I^{\mathrm{bi}}(R).$$

Note first that $Q_I^{\mathrm{bi}}(R) \subset R_P$ for all $I \subset P$. Indeed, if $q \in Q_I(R)^R$, then we may find a left ideal $L \in \mathcal{L}(\sigma^I)$ with $Lq \subset R$. Since $LRq \subset R$ and $LR \in \mathcal{L}(\sigma^I)$ as well, we may assume L to be two-sided. It follows that the R-bimodule $R + Rq/R \subset Q_I^{\mathrm{bi}}(R)/R$ is annihilated by L. But $L \in \mathcal{L}^2(\sigma_P)$, for if $L \subset P$, then

$\mathrm{Ann}_{R/P}(L) = R/P \neq 0$, whereas $\mathrm{Ann}_{R/P}(I) = 0$, since $I \not\subset P$. We thus have that $R + Rq/R$ is σ_P torsion, hence $q \in Q_{\sigma_P}(R + Rq) = R_P$, showing that $Q_I^{\mathrm{bi}}(R) \subset R_P$. Conversely, since R_P/R is a σ_P-torsion bimodule, for any $q \in R_P^R$ we may find $I \in \mathcal{L}^2(\sigma_P)$, i.e. $I \not\subset P$, with $Iq \subset R$, hence $q \in Q_I(R)$, and even $q \in Q_I(R)^R$. It follows that $R_P^R \subset \cup Q_I(R)^R$ hence that $R_P \subset \bigcup Q_I^{\mathrm{bi}}(R)$, applying [13, V. 3.7] to the inductive union of the $Q_I^{\mathrm{bi}}(R)$. This proves the assertion. □

Note that from this or the fact that $\sigma^I = \sigma_I$ for any finitely generated ideal in a commutative ring, one easily deduces that $\boldsymbol{O}_R$ is just the usual structure sheaf on Spec(R) in the commutative case.

11. PROPOSITION. *The global sections of $\boldsymbol{O}_R$ are given by* $\Gamma(\mathrm{Spec}(R), \boldsymbol{O}_R) = R$.

PROOF. Consider the etale space $\boldsymbol{O}_R$ on Spec(R) associated to $\boldsymbol{O}_R$. A basis of open sets for $\boldsymbol{O}_R$ may then be given by the $\tilde{s}(I)$, which consist of the families of $s_P \in R_P$, where $s \in Q_I^{\mathrm{bi}}(R)$, where s_P is just viewed as an element of R_P and where P varies through $X(I) \subset \mathrm{Spec}(R)$. It follows that a global section of $\boldsymbol{O}_R$ or Q_R is given by a family of open subsets $X(I_\alpha)$ which covers Spec(R) and an element

$$s \in \bigcap_\alpha Q_{I_\alpha}^{\mathrm{bi}}(R).$$

But then $\Sigma I_\alpha = R$, hence

$$\bigcap_\alpha Q_{I_\alpha}^{\mathrm{bi}}(R) \subset \bigcap_P R_P.$$

Consider an element

$$s \in \bigcap_P R_P,$$

then we may find for each prime ideal P an ideal $I_P \not\subset P$ with $I_{P^s} \subset R$, as one easily verifies, hence

$$\left(\sum_P I_P\right) s \subset R,$$

so $s \in R$, since $(\Sigma I_P) \not\subset P$ for all $P \in \mathrm{Spec}(R)$. It follows that $\Gamma(\mathrm{Spec}(R), \boldsymbol{O}_R) \subset R$. This other inclusion is obvious, since $\boldsymbol{Q}_R(\mathrm{Spec}(R)) = R$ and $\boldsymbol{Q}_R$ is separated.

□

12. COROLLARY (of the proof). *For any open subset* $X(I) \subset \mathrm{Spec}(R)$ *we have*

$$\Gamma(X(I), O_R) = \bigcap_{I \subset P} R_P.$$

This may be proved along the lines of the previous proof. □

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